The Third Pair

Encounters with cubes

1. It was in my first visit to Dr. Koelbel that I became aware, once again, of cubes.

Dr. Koelbel, who treats residents here at The Boston Home, is a specialist in the Baclofen pump. Baclofen is the anti-spasticity drug for people with MS that is usually taken orally. But this has to go through the digestive system, and it tends to make the patient sleepy and weak. By contrast, the pump, which is implanted into one's left side, delivers Baclofen directly to the central nervous system. So it is much more efficient. Whereas with oral doses one measures in milligrams, with the pump measurement is in micro-grams.

The Baclofen pump must be refilled every four or five months: this is done by needle through one's skin. To have the refill, I would be transported by chair van to Dr. Koelbel's office in Braintree.

When I reached the door to his office, I noticed the number was '216'. That number immediately jumped out at me--I knew it was significant, but I couldn't yet remember how.

And then I recalled our games of backgammon.

My daughter Gretchen and I used to play backgammon nearly every day. In that game, the first person to throw off all of his/her pieces wins. But we didn't want to accept such sudden finality; rather, we were willing to give that other person 'last licks' in the form of one more dice throw.

But that wasn't all: suppose that that would-be loser still could not tie the game on one dice throw? Even then there was still hope: if they threw a double, they were allowed to throw again.

There are times in backgammon when one is in such a woeful position vis-à-vis the potential winner, that only the throwing of three doubles in a row will rescue the game.

Why do I remember all this business of three doubles? Because once in a backgammon game, when I was woefully behind and I had to throw three doubles in a row just to rescue a tie, I did just that (to the consternation of my opponent, I can assure you!)

So how difficult is it to throw three doubles in a row? (in mathematical terms, what is the probability?).

We first figure out the chances of throwing one double. There are six possible doubles out of a total of 36 different combinations of the dice. So the probability is $^6/_{36}$ or $^1/_6$.

To find the chances for doubles on multiple (independent) throws of the dice, you multiply the individual probabilities together. So the probability of throwing three doubles in a row is: $1/6 \bullet 1/6 \bullet 1/6 = 1/216$.

What kind of odds are these? It means that, if you throw a large number of throws (in the hundreds of thousands), you will throw three doubles in a row on an average of about every 216 throws.

Here's another way to look at it. There is a chain of Italian restaurants in the Boston area called Vinny Testa’s. Each table at these restaurants has a glass jar filled with small cork disks numbered from 1 to 200. Likewise, the checks are numbered from 1 to 200. At the end of the meal, the waitress draws a disc from the jar. If the number on the disc corresponds to the number on your check, the dinner is on the house. The odds of getting a free dinner at Vinny Testa’s is about the same as that of throwing three doubles in a row.

Yet, I know some people who once got free meals at Vinny Testa's despite those odds because their number was drawn.

Anyway, I had an "aha!" moment at Dr. Koelbel's, when I recognized that his office number of '216' was a perfect cube — that $216 = 6 \bullet 6 \bullet 6 = 6^3$. When I told him that, not only was he not clueless or bored, he actually was interested! It turns out he had been a math minor in college, and so he had an appreciation for such mathematical *arcana*.

2. My discussion with Dr. Koelbel reminded me of a story concerning the great Indian mathematician Ramanujan, and I related it to him:

"English mathematician G. H. Hardy had brought Ramanujan to England to work with him at Trinity College in Cambridge. Unfortunately, Ramanujan became ill with consumption due to the English climate. One time when he was in the hospital, Hardy went to see his colleague.

"Ramanujan asked Hardy if he had seen any interesting numbers on the way over. Hardy replied, 'Not really. My cab number was 1729, but I don't consider that to be of any importance.' Ramanujan retorted, ‘On the contrary: that is a very interesting number! It is, in fact, the smallest number that can be expressed as the sum of two cubes in two different ways.’" Indeed,

$(12)^3 + (1)^3 = 1728 + 1 = 1729$. And,

$(10)^3 + (9)^3 = 1000 + 729 = 1729$.

(A few years ago this number arose for me in quite a different context. One summer when we were at Lake George, the weekly bill came for the lodge in which our extended family was staying. It was $1729. When I told my relatives about the significance of this number ("the smallest number that can be expressed as the sum of two cubes in two different ways"), most of them had no idea what I was talking about. But one of my brothers-in-law who was an engineer (but in this one instance was behaving like a mathematician) said, "Prove it!" I couldn't prove that it is the smallest such number other than by trial and error and exhaustion (which as a 'proof' for a mathematician is 'inelegant'.) I also have no idea whether or not there is a larger such number, never mind demonstrate one.)

3. That night in bed (and for a few nights after that), in order to keep my mind exercised and to counter boredom, I began to calculate cubes of numbers larger than 12. Since I was doing these calculations in my head, I used little tricks to make my work easier. For example, it is very easy to multiply by 10 or 100. So, using simple algebraic techniques such as the distributive law, I would divide a problem up into sub-problems. So for example:

$$\begin{aligned}(13)^2 &= (13)(13) = (10+3)(10+3)\\ &= (10)(10) + (10)(3) + (3)(10) + (3)(3)\\ &= 100 + 30 + 30 + 9\\ &= 169\end{aligned}$$

Likewise,

$$\begin{aligned}(13)^3 &= (13)(13)(13) = (169)(13) = (169)(10+3)\\ &= (169)(10) + (169)(3)\\ &= 1690 + (170-1)(3)\end{aligned}$$

etcetera

Then, once having calculated such cubes, I development mnemonics to remember them. Some of these were personal facts. For example,

$(13)^3$ = 2197 (our daughter Gretchen was 21 years old in 1997.) And,
$(21)^3$ = 9261 (my grandfather was 92 years old in 1961.)

Others I remembered, at least in part, because of mathematical coincidence. For example, the cubes of 24 and 25 end narcissistically with themselves:

$(24)^3$ = 13824.
$(25)^3$ = 15625.

There are some which contain mathematical facts. For example,

$(18)^3$ = 5832 : here all the numerals, including those of the original number and the very cube number, are elements of the Fibonacci sequence (1, 1, 2, 3, 5, 8,... wherein each number is obtained by adding the previous two.)

Thus did I calculate and memorize--up to and including the cube of 27, where things became too onerous. I then checked my answers by dividing (by long division) twice by the original number to obtain that number. Thus for example, I had calculated that $(26)^3 = 17,576$ by multiplying. So to check, we divide: 17,576 ÷ 26 = 676. And 676 ÷ 26 = 26.

I did find that $(27)^2 = 729$, which I recognized as an old friend. So it seems that $(9)^3 = (27)^2$. Actually, this equality can be verified without doing the individual calculations -- by using laws of exponents and expressing them each as a power of the same number:

$729 = (9)^3 = ((3)^2)^3 = (3)^6 = ((3)^3)^2 = (27)^2 = 729$.

4. A couple of weeks later, again in bed at night, I began thinking once more upon Ramanujan and the sums of cubes. I recalled from basic algebra the formula to factor the sum of two cubes:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Substituting one pair of Ramanujan's numbers, we have.

$$(12)^3 + (1)^3 = (12 + 1)(12^2 - (12)(1) + 1^2) = (13)(144 - 12 + 1) = (13)(133) = 1729$$

Thus, 13 is a factor (i.e., divisor) of 1729. Likewise,

$$(10)^3 + (9)^3 = (19)(91) = 1729$$

and so 19 is also a divisor of 1729.

We know that 13 and 19 are prime numbers.

If we divide 1729 successively by 13 and 19, we find out the third prime factor: 7. So 1729 = (7)(13)(19).

And then it hit me-- one of those "eureka!" moments that mathematicians pray for (or would if they prayed): since the other two pairs of numbers added to two of the prime factors, there must be a third pair of numbers which add to the third factor (7), and the sum of whose cubes is 1729!

It occurs to me now, that that sort of thinking was decidedly hubristic. After all, if this were true, why hadn't Ramanujan thought of it? In my conceit, I decided that he had somehow overlooked it.

True, it seemed unlikely that the two numbers would both be positive (I checked a few examples just to make sure.) I then considered the possibility that the two were still whole numbers, but that one of them was negative (the cube of a negative number is negative.) So I tried a few pairs-- for example, 18 and (-11); but none of these results were anywhere near 1729.

At that point, stubbornly, I decided to solve the problem analytically. Recalling once again the factorization of two cubes -- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ -- I let x and y represent the two numbers. Beginning with $x + y = 7$ and substituting $y = 7 - x$, we get

$(7)(x^2 - x(7 - x) + (7 - x)^2) = 1729.$
Dividing by 7, multiplying out, and collecting like terms, we have
$3x^2 - 21x + 49 = 247.$ Subtracting and then dividing by 3, we get
$x^2 - 7x - 66 = 0.$

I checked the discriminant and found out that, unfortunately, this quadratic equation is not factorable; and therefore, its roots are not rational numbers. My dreams for another pair of whole numbers were dashed. Nevertheless, I decided to solve the equation using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - (4)(1)(-66)}}{2}$$

The solutions are: $x = \dfrac{7 + \sqrt{313}}{2}$ *and* $\dfrac{7 - \sqrt{313}}{2}$

These solutions are irrational. (Rational approximations are 12.34 and (-5.34). This shows that my hunch was correct about one solution being negative.) Should we check them? Of course. When you add them together, the square roots cancel out and the result is $\frac{7}{2} + \frac{7}{2} = 7.$

We now need to cube each number and add them together. This seems formidable; but it helps to realize that each number is essentially a binomial, and then to use the formulas for cubing a binomial:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Again the square roots cancel and, incredible as it may seem, the sum of these two numbers cubed is exactly 1729.

So did I beat Ramanujan? Of course not. When he said "the sum of two cubes", he was speaking of whole positive numbers. As for my mathematical instincts, they did not fail me; I just didn't get the kind of solutions I was hoping for.

5. By the way, my most recent appointment with Dr. Koelbel was a few months ago. When I arrived at the medical building, I was told that his office had been moved to the third floor. Once there, I sat in the waiting room and stared at the office number printed on the door: it was '343'.

And I began to calculate...

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