Timely Encounter

A geometry of the clock

(This is an abridged version of Hospital Clock)

By Theodore May

Recently I was in the hospital for five days due to a fever. Because the doctors were not sure as to the cause of the fever, I was placed in a private room. This room had a small ancient TV which gave a horrible picture; but as I don't watch TV, that did not bother me. I was so weakened by the fever that I could not lift my arms from the bed (so no reading, never mind getting up or moving about). I was, you might say, trapped in that bed in that room for five days.

Directly in front of me on the opposite wall there was an analog clock. After awhile, I began to devise geometry problems associated with the clock. Little by little I felt myself drawn into a fascinating little mathematical universe. In those five days I did not come close to exhausting the kinds of problems one can encounter in this closed little system. I and others continue to discover new things more than a year and a half later.

The only mathematics involved in the investigation that follows is the use of fractions and simple algebraic equations.

Lucky is the person who has such a hospital stay as this!

On the First Day, I familiarized myself with the basic relations on the clock:

- The minute hand moves 12 times as fast as the hour hand. (Thus, the hour hand moves one 'notch' when the minute hand moves 12 of them);
- The angle between two successive numbers is 30 degrees (this is 1/12 of 360); and thus each minute is 6 degrees of arc.

I also found approximations for certain values, thus familiarizing myself with the problem it presented:

<u>EX</u>: Estimate where the hands coincide after 4:00.

<u>SOLN</u>: It cannot be as late as 4:24, for then the minute hand is beyond the hour hand (the hour hand has moved exactly two notches.) So the time must be a bit less than 4:22.

<u>Problem 1 for the Reader</u>: Estimate the time where the hands coincide after 9:00.

(Answer: Just a bit after 9:49)

On the Second Day: Coincidence problems.

I did these by treating them as problems of pursuit such as are found in elementary algebra texts. In this case the minute hand is chasing after the hour hand.

<u>EX</u>: Find exactly when the two hands coincide after 4:00.

<u>SOLN</u>: Let x = distance (in minutes) Hour Hand moves then 12x = distance (in minutes) Minute Hand moves

Position Minute Hand after 4:00 = Position Hour Hand after 4:00

The Hour Hand has a 20-minute head start

$$12x = 20 + x$$

$$11x = 20$$

$$x = \tfrac{20}{11} = 1\tfrac{9}{11}$$

$$12x = 12 + \tfrac{108}{11} = 21\tfrac{9}{11}$$

So the hands coincide at exactly <u>4:21 9/11</u>

(Compare this result to the approximation above.)

All Coincidence problems have this basic form.

Problem 2: Find exactly when the two hands coincide after 9:00.

(Ans: 9:49 1/11)

Problem 3: Calculate exactly when the two hands coincide after 11:00.

(Yes, I know, the answer is obvious. But calculate it anyway, as it shows that these methods work for extreme cases.)

(Ans: 12:00)

It has been called to my attention by my cousin Dan (who also provided the clock diagrams) that the results of coincidence problems can be expressed much more simply and elegantly in terms of hours alone. Thus, e.g., the answer to the problem, "When are the hands coincident after 4:00?" can be rewritten thus:

$$4:21\tfrac{9}{11} = 4 + \tfrac{21}{60} + \left(\tfrac{9}{11}\right)\left(\tfrac{1}{60}\right) = 4 + \tfrac{231}{660} + \tfrac{9}{660}$$

$$= 4 + \tfrac{240}{660} = 4 + \tfrac{4}{11}\,hours$$

This tempts us to offer the following:

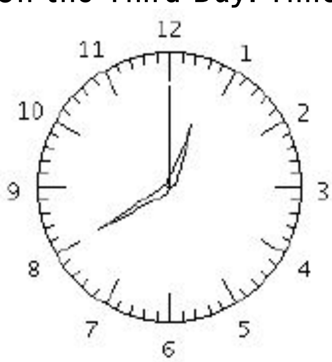
THEOREM 1: Let n be a number on the clock. Then the time after n-o'clock at which the hands are coincident is n + n/11 hours.

Proof: As we have seen above, such solutions begin with an equation of the form 12x = 5n + x (where everything is in minutes.) So x = 5n/11, and thus 12x = 60n/11. To convert this to hours, divide by 60. Thus the time of coincidence is n + n/11 hrs. QED

Problem 4: At what time after 9:00 are the hands coincident? Express your answer in hours, and then convert to minutes.

(Answer: 9 + 9/11 hrs; 9:49 1/11)

On the Third Day: Time-Angle problems. Given the Time, what is the Angle?



EX 1: What angle do the hands make at 12:40?

SOLN: The Minute Hand is 2/3 around the hour. So the Hour Hand must be 2/3 from 12 to 1. Since the whole angle in the latter is 30 degrees, 2/3 of 30 is 20. So the angle between the Hour Hand and 1 is 10 degrees.

If the Hour Hand were on 2, the hands would form 180 degrees. The angle from 1 to 2 is 30 degrees. So the angle at 12:40 is 180 - 30 - 10 = 140 degrees.

EX 2: Find the angle of the hands at 12:41.

SOLN: Compare this to the above:

The Minute Hand has moved 6 degrees.

So the Hour Hand has moved $\frac{1}{12}(6) = \frac{1}{2}$ degree.

So the angle has decreased by $6 - \frac{1}{2} = 5\frac{1}{2}$ degrees.

Thus at 12:41 the angle is $140 - 5\frac{1}{2} = 134\frac{1}{2}$ degrees.

The following Theorem can be readily verified using the reasoning of the above problem.

THEOREM 2: A change in one minute of time causes a change of $5\frac{1}{2}$ degrees in the angle between the hands.

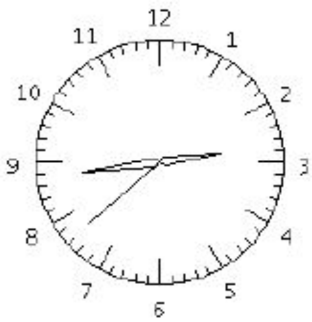
Problem 5: a) Find the angle between the hands at exactly 5:00.

b) Using the Theorem, extrapolate to find the angle at 5:06.

(Answers: 150 degrees; 117 degrees)

On the Fourth Day: Angle-Time problems. Given the Angle, what is the Time?

(Note: For a long time I was mired in trying to find solutions to these problems using degrees. The breakthrough came when I converted degrees to minutes, and saw them as other types of pursuit problems such as those found on Day 2.)



EX 1: At what time after 2:00 are the hands in opposite directions (i.e. make a straight angle)?

SOLN: We note that the answer is a bit beyond 2:43.

A straight angle is 30 minutes.

Amount Minute Hand moves = 10 + Amount Hour Hand moves + 30. (The Hour Hand has a head start of 10 minutes.)

$$12x = 10 + x + 30$$

$$11x = 40$$

$$x = \tfrac{40}{11} = 3\,\tfrac{7}{11}$$

$$12x = 12\left(3\,\tfrac{7}{11}\right) = 36\,\tfrac{84}{11} = 43\,\tfrac{7}{11}$$

So the hands form a straight angle at 2:43 7/11.

Problem 6: Convert this time to hours.

(Answer: 2 + 8/11)

It is intriguing to speculate on how these latter numbers relate to a straight angle. A little experimentation with other angles yields some data following the definition:

DEFINITIONS: A *fore angle* is an angle wherein the minute hand is ahead of the hour hand. Likewise, an *aft angle* has the minute hand behind the hour hand. (Angles are between 0 and 180 degrees inclusive.)

Times after 2:00 the angle is:

30 deg. Fore angle: 2 + 3/11 hours; Aft: 2 + 1/11 hours.
45 deg. Fore: 2 + 3.5/11 hours; Aft: 2 + .5/11 hours.
60 deg. Fore: 2 + 4/11 hours; Aft: 2 + 0/11 [2:00]

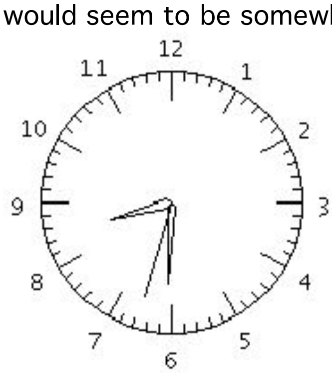
This suggests the following theorem:

THEOREM 3: Let p be a number between 0 and 6. Then the time in hours that the hands make an r-degree angle after n-o'clock is n + (n +/- p)(mod 12)/11, where r = 30p. (The signs are chosen as to whether the angle is fore or aft respectively.)

Proof: The equations are: 12x = 5n + x + 5p and 12x + 5p = 5n + x. These combine to: 12x = 5n + x +/- 5p (the latter being the number of minutes in the angle.) The solutions are x = (5n +/- 5p)/11 = 5(n +/- p)/13. So 12x = 60(n +/- p)/11 is the total minutes after n-o'clock. This in hours is (n +/- p)/11, and so the theorem follows. QED

EX 2: At what time after 8:00 do the hands form an angle of 72 degrees?

SOLN 1 (by pieces): We note that 8:27 is very near 90 deg. So the answer to this problem would seem to be somewhere around 8:30.



Recall that there are 6 degrees for each minute.

So 72 degrees = 72/6 minutes = 12 minutes.

Amount Minute Hand moves + 12 = 40 + Amount Hour Hand moves

$$12x + 12 = 40 + x$$

$$11x = 28$$

$$x = \tfrac{28}{11} = 2\,\tfrac{6}{11}$$

$$12x = 12\left(2\,\tfrac{6}{11}\right) = 24\,\tfrac{72}{11} = 30\,\tfrac{6}{11}$$

So the hands are 72 degrees apart at exactly 8:30 6/11.

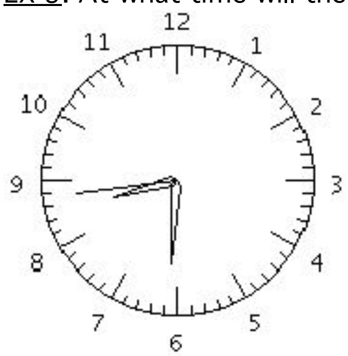
SOLN 2 (Using the above Theorem):

The problem implies that the minute hand is behind the hour hand – that is, that the angle is an aft angle. So we choose the negative sign.

Here 72 = 30p. So p = 72/30 = 2.4

So the time is 8 + (8 - 2.4)/11 = 8 + 5.6/11 hrs.

EX 3: At what time will the above hands reach 71 degrees?



SOLN: The equation is almost the same.

6 degrees = 1 minute

1 degree = $\frac{1}{6}$ minute

$$12x + 11\tfrac{5}{6} = 40 + x$$

$$11x = 28\tfrac{1}{6}$$

$$x = \tfrac{28}{11} + \tfrac{1}{66} = 2\tfrac{6}{11} + \tfrac{1}{66} = 2\tfrac{36}{66} + \tfrac{1}{66} = 2\tfrac{37}{66}$$

$$12x = 12\left(2\tfrac{37}{66}\right) = 24 + 12\left(\tfrac{37}{66}\right) = 24 + 2\left(\tfrac{37}{11}\right) = 24 + \tfrac{74}{11} = 24 + 6\tfrac{8}{11} = 30\tfrac{8}{11}$$

So the hands are 71 degrees apart at 8:30 8/11

Notice that a change of 1 degree in angle caused a change of $\frac{2}{11}$ minutes in time. This seems to always be the case and it can stated as a theorem:

THEOREM 4: A change of 1 degree in angle = A change of $\frac{2}{11}$ minutes in time.

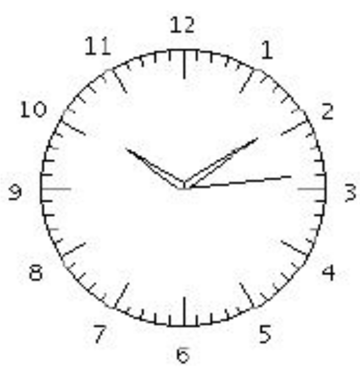
Problem 7: By extrapolation using the information given above, find the exact time after 8:00 when the hands form an angle of 64 degrees.

(Answer: 8:32)

On the Fifth Day: Symmetry Problems.

There are twelve symmetries about each of six lines. (Note: The term 'symmetry about the number n' means the same as 'symmetry about the n-(n +/- 6) axis.') (The minus sign is for n>6.)

a) Symmetries about the vertical (12-6 symmetry)



EX 1: Find the symmetry near 10:10

SOLN: (This is the time pictured in every watch ad. It is sometimes called 'ten-ten symmetry'.)

Amount change on left above 10 = Amount change on right above 2

$$x = 10 - 12x$$

$$13x = 10$$

$$x = \tfrac{10}{13}$$

$$12x = \tfrac{120}{13} = 9\tfrac{3}{13}$$

So the exact time of symmetry is 10:09 3/13.

(Note that each hand is 10/13 from its nearest number.)

Problem 8: Convert the above result to hours.

(Answer: 10 + 2/13 hrs.)

A bit more investigation yields this result:

THEOREM 5: The time after n-o'clock for symmetry about the 12-6 axis is given by n + (12 – n)/13 hours.

Proof: Let x = Amt (in minutes) hour hand moves. Since (12 - n) is the symmetrical opposite of n, the equation is given by x = 5(12 - n) – 12x. So x = 5(12 - n)/13. Thus 12x = 60(12 – n)/13 is the minutes after n-o'clock; this converts to (12 - n)/13 hours, and the theorem follows. QED

The following example illustrates the next theorem.

EX 2: Show that the two time-angles 10 + 2/13 hrs. and 1 + 11/13 hrs. are coincident.

SOLN: For the first, the minute hand is 2/13 hrs. away from '12', and this may be taken as a 'measure' of half of the angle. For the second angle, the minute hand is 11/13 around the hour, and thus the 'measure' of the half angle is 1 – 11/13 = 2/13. Note that the times add to 12:00.

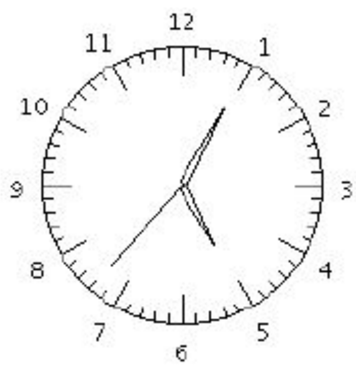
As far as I can tell, the 12-6 symmetry angles are the only ones having this property. I call the two angles 'reflection angles'.

DEFINITION: *Reflection angles* are two time-angles which are coincident.

THEOREM 6: Let A be a time-angle symmetric about the 12-6 axis. Then there is another time-angle A' coincident with A. Further, the times of A and A' add to 12:00.

Problem 9: Find the reflection angle of 7 + 5/13 hrs. Show the two angles are coincident.

(Ans: 4 + 8/13 hrs.)



b) Symmetries about the horizontal (3-9 symmetry)

EX 3: Find the symmetry near 5:05.

SOLN:

$$x = 5 - 12x$$

$$13x = 5$$

$$x = \tfrac{5}{13}$$

$$12x = \tfrac{60}{13} = 4\ \tfrac{8}{13}$$

So the exact time of symmetry is <u>5:04 8/13</u>.

(The hour time is 5 + 1/13.)

There are four other lines of symmetry and their corresponding times.

Problem 10: Find the exact time for the symmetry near 1:15 about the 2-8 line.

Problem 11: Find the angle formed by the hands at ten-ten symmetry (i.e. at exactly 10:09 3/13.)

(Ans: 1:13 11/13 or 1 + 3/13 hrs; 110 10/13 degrees.)

The hourly answer to problem 2 suggests this:

THEOREM 7: Let m and n be two numbers on the clock such that (m + n)/2 = q (an integer). Then the hourly time-angle m + n/13 is symmetrical about q (i.e. the q-(q +/- 6) axis.)

Proof: By definition, q is the midpoint between m and n. So m and n are symmetrical about q. We will prove that the hour- and minute-hands of m + n/13 are the same but opposite distances from m and n respectively, and thus symmetrical about q as well.

For the minute hand, we note that the number n is 5n minutes = 5n/60 hrs. = n/12 hrs. So
Distance minute hand from n (in negative direction)
= n/12 – n/13 = 13n/156 – 12n/156 = n/156 hrs.

For the hour hand, we note that it moves at 1/12 the speed of the minute hand. So
Distance hour hand from m (in positive direction)
= (1/12)(n/13) = n/156 hrs. QED

Problem 12: Find some hour angle-times symmetrical about the 4-10 axis.

(Ans: Samples: 2 + 6/13, 6 + 2/13, 4 + 4/13, 7 + 1/13, 9 + 11/13, etc.)

DEF: The fractional part of the hourly time for a time-angle is called its *DNA*. So for example, if the hourly time is $8 + \frac{3}{11}$, then the hour DNA is $\frac{3}{11}$. You can also obtain the minute DNA by multiplying by 60 ($\frac{180}{11}$ in our example).

A *Family* consists of all those time-angles which have the same DNA. For non-symmetrical angles, the n-Family is that family all of whose members have hour-DNA = $\frac{N}{11}$. For symmetrical angles the hour DNA=$\frac{N}{13}$.

The following theorem will be useful for the next one:

THEOREM 8: (a) Let 'n' be a number on the clock. Then the minute-DNA of the non-symmetrical n-Family = 5n + 5n/11.

(b) The minute-DNA of a symmetrical n-family = $4n + \frac{8n}{13}$

Proof: Since the minute-DNAs are $\frac{60n}{11}$ and $\frac{60n}{13}$ the results follow by long division. QED

(In this section I will not give units on hourly times when the meaning is obvious.)

DEF: The smallest aft angle symmetrical about a point is called an *aft sliver angle*. Likewise, the smallest fore angle symmetrical about a point is called a *fore sliver* angle.

THEOREM 9: Let n be a number on the clock. Then

Time aft sliver angle at n = n + n/13 hrs.

Time fore sliver angle at n = (n – 1) + (n + 1)/13 hrs.

Proof: If m and n are any two numbers on the clock, then by Theorem 7, an angle of the form m + n/13 is symmetric about the number q = (m + n)/2 provided q is an 1nteger. So n + n/13 is symmetric about n. Further, it is an aft angle, since the hour hand is beyond symmetry point n. And: it is clearly the smallest such angle.

For the fore angle, the hour hand cannot be beyond symmetry point n. So the hour must be (n – 1). But then, by symmetry, its DNA must be (n + 1)/13. QED

Sometimes things are not what they at first seem:

EX 2: Find the time of symmetry about 9 near five minutes before 7.

SOLN: Our first guess might be 7 + 11/13 hrs. But this cannot be, for then the hour hand would be near 8. We conclude that, perhaps contrary to intuition, it must be 6 + 12/13 hrs. Indeed, this gives a minute time of 6:55 5/13, as desired.

Problem 13: Consider the aft angles of the symmetrical seven-family:

(a) Find the minute-DNA of that family.

(b) Find the times for symmetry about vertical, horizontal, and the 1-7 axis. (The latter is the aft sliver angle.)

(c) Find the measurement of the latter sliver angle.

(Ans: (a) 32 4/13 (or 7/13 hrs.)

(b) 5:32 4/13, 11:32 4/13, 7:32 4/13,

(c) 32 4/13 degrees)

Wow! Compare the answers to (a) and (c): is that a coincidence? It isn't, as the theorem below shows.

ANGLE-TIME THEOREM (ATT): For any symmetrical n-family, Measure aft sliver angle at n = Minute-DNA n-family.

Proof: Starting at n-o'clock, as the minute hand moves toward the sliver angle, the hour hand moves 5n/13 minutes. This is half of the symmetrical angle; so the whole angle is 10n/13 min. But since there are six degrees for every minute,

Measurement aft sliver angle = 6(10n/13) degrees

= 60n/13 = Minute-DNA aft n-family. QED

Problem 14: Use Theorem 10 and the ATT to find:

- the time and measure of the aft sliver angle about 3.
- the time of the fore sliver angle about 3.

(Ans: 3:13 11/13, or 3 + 3/13 hrs.; 13 11/13 degrees;

2:18 6/13, or 2 + 4/13 hrs.)

COROLLARY 1:

The measure of any aft symmetric angle m + n/13 = 30(m – n) + Minute-DNA n-family.

(NB: If the angle spans across '12', then 12 should be added to the value for m.)

Problem 15: Find the measure of the angle 7 + 3/13.

(Ans: 133 11/13 degrees)

A bit of investigation turns up the fact that the aft sliver angles increase in size arithmetically from n=1 through n=12, while the fore sliver angles decrease by the same amount over that interval. Further, problems like the following yield more information:

Problem 16: (a) Find the times for the fore and aft symmetrical sliver angles and their measures about 2.

(b) Find the times for the fore and aft symmetrical sliver angles and their measures about 10.

(Solns: (a) Fore: 1:13 11/13 (1 + 3/13 hrs.), 46 2/13 deg;

Aft: 2:09 3/13, (2 + 2/13 hrs.), 9 3/13 deg.

(b) Fore: 9:50 11/13, (9 + 11/13 hrs.), 9 3/13 deg;

Aft: 10:46 2/13, (10 + 10/13 hrs.), 46 2/13 deg.)

We note that the size of the fore angle of one is equal to the size of the aft angle in the other. This is true for these sorts of angles in general.

DEF: Two symmetric sliver angles, one fore and the other aft, associated with the numbers 'n' and '12 - n' on the clock are called *dual angles*.

THEOREM 10: Let A(n) and B(12 - n) be aft and fore dual sliver angles respectively about n and (12 – n). Then Measure A(n) = Measure B(12 - n).

Proof: For aft angle A(n), we have already seen that the 'measure' of half the angle in minutes is 5n/13. We need to show this true for fore angle B(12 - n) as well.

To produce the fore angle, we begin at (12 – n – 1)-o'clock. The minute hand moves, passes the hour hand, and winds up on the other side of (12 – n). The hour hand sweeps out 5(12 – n + 1)/13 minutes, but this is outside the angle. To get the part in the angle, we must subtract from 5:

'Measure' half fore sliver angle B(12 - n) in minutes

$$= 5 - \frac{5(12-n+1)}{13} = \frac{65}{13} - \frac{5(13-n)}{13} = \frac{65}{13} - \frac{65-5n}{13}$$

$$= \frac{65}{13} - \frac{65}{13} + \frac{5n}{13} = \frac{5n}{13} = \frac{5n}{13}\text{ min.} \quad QED$$

<u>Problem 17</u>: a) Using Theorems 9, 10, and the ATT, find the measure of the fore sliver angle which is the dual of the aft sliver angle at 9.

b) Find the times of the two angles.

(Ans: a) 41 7/13 degrees. b) Aft: 9:41 7/13, or 9 + 9/13 hrs.;

Fore: 2:18 6/13, or 2 + 4/13 hrs.)

THEOREM 11: Let A(n) and B(12 - n) be dual sliver angles.
Then their times add to 12:00.

Proof: By definition of 'dual' and by Theorem 10, the two time-angles can be written

A(n) = (n – 1) + (n + 1)/13 (fore), and

B(12 – n) = (12 – n) + (12 – n)/13 (aft).

Then the sum of their times

$$= n - 1 + \tfrac{n}{13} + \tfrac{1}{13} + 12 - n + \tfrac{12}{13} - \tfrac{n}{13} = 11 + \tfrac{13}{13} = 12 \quad QED$$

COROLLARY 2: The Minute-DNAs of two dual angles add to 60.

Problem 6: Using the above, find the time for the dual of the sliver angle at 3:23 1/13.
Then use the ATT Theorem to find the measure of the two angles. (Which one is the aft angle?)

(Ans: 8:36 12/13; 36 12/13 degrees)

I've included an exam in the appendix which will allow the reader to evaluate his/her skills.

Acknowledgements: I would like to thank Dan Budne, Phil Lewis, and Leslie Lewis for their help, encouragement, and suggestions for improvement.

Appendix

Chronological Test: All times are in order from 8:00 to 9:00. You may express times in minutes or hours – or both! Answers below. (*extra credit)

Find:

1) Measure of angle at 8:00.
2) Time after 7:00 for fore angle symmetry about 10.
3) Time for symmetry about 11.
4) Measure of angle at 8:10.
5) Time for straight angle.
6) Time for 179-degree angle after above.
7) Time for aft 150-degree angle.
8) Time for symmetry about 6.
9) Time for aft 90-degree angle.
10) Time for symmetry about 7.
11) Measure of angle at 8:30.
12) Time for aft 60-degree angle.
13) Time for aft sliver angle at 8.
14) Measure of aft sliver angle at 8.
15) Time for coincidence of hour/minute hands.
*16) Angle measure at 8:44.
17) Time for fore sliver angle at 9.
18) Measure of fore sliver angle at 9.
19) Time after 8:00 for fore angle symmetry about 10.
20) Measure of angle at 8:59.
21) Time for fore 90-degree angle.

Answers:

1) 120 degrees.
2) 8:00, 7 + 13/13 hrs.
3) 8:09 3/13, 8 + 2/13 hrs.
4) 175 degrees.
5) 8:10 10/11, 8 + 2/11 hrs.
6) 8:11 1/11, 8 + 2/11 + 1/330 hrs.
7) 8:16 4/11, 8 + 3/11 hrs.
8) 8:18 6/13, 8 + 4/13 hrs.
9) 8:27 3/11, 8 + 5/11 hrs.
10) 8:27 9/13, 8 + 6/13 hrs.
11) 75 degrees.
12) 8:32 8/11, 8 + 6/11 hrs.
13) 8:36 12/13, 8 + 8/13 hrs.
14) 36 12/13 degrees.
15) 8:43 7/11, 8 + 8/11 hrs.
*16) 2 degrees.
17) 8:46 2/13, 8 + 10/13 hrs.
18) 13 11/13 degrees.
19) 8:55 5/13, 8 + 12/13 hrs.
20) 84.5 degrees.
21) 9:00, 8 + 11/11 hrs.